ECS 315: Probability and Random Processes **2017/1**

HW 12 — Due: Not Due

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Problem 1 (Yates and Goodman, 2005, Q3.3.4)**.** The pdf of random variable Y is

$$f_Y(y) = \begin{cases} y/2 & 0 \le y < 2, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find $\mathbb{E}[Y]$.

(b) Find $\operatorname{Var} Y$.

Problem 2 (Yates and Goodman, 2005, Q3.3.6)**.** The cdf of random variable V is

$$F_V(v) = \begin{cases} 0 & v < -5, \\ (v+5)^2/144, & -5 \le v < 7, \\ 1 & v \ge 7. \end{cases}$$

(a) What is $f_V(v)$?

(b) What is $\mathbb{E}[V]$?

(c) What is $\mathrm{Var}[V]$?

$\mathbb{E}[g(V)] = \int_{-\infty}^{\infty} g(v) f_V(v)\, dv$

(d) What is $\mathbb{E}[V^3]$?

Problem 3 (Yates and Goodman, 2005, Q3.4.5)**.** X is a continuous uniform RV on the interval $(-5, 5)$.

(a) What is its pdf $f_X(x)$?

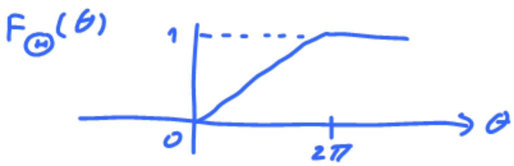
(b) What is its cdf $F_X(x)$?

(c) What is $\mathbb{E}[X]$?

(d) What is $\mathbb{E}[X^5]$?

(e) What is $\mathbb{E}\left[e^X\right]$?

$\int_{-\infty}^{\infty} f_{\Theta}(\theta)\, d\theta = 1$



Problem 4 (Randomly Phased Sinusoid)**.** Suppose Θ is a uniform random variable on the interval $(0, 2\pi)$.

$f_{\Theta}(\theta) = \begin{cases} 1/2\pi, & 0 < \theta < 2\pi, \\ 0, & \text{otherwise}. \end{cases}$

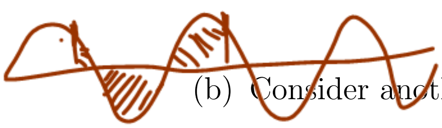
(a) Consider another random variable X defined by

$$X = 5\cos(7t + \Theta) = g(\Theta)$$

$g(\cdot) = 5\cos(7t + (\cdot))$

where t is some constant. Find $\mathbb{E}[X]$.

$\mathbb{E}[X] = \mathbb{E}[g(\Theta)] = \int_{-\infty}^{\infty} g(\theta) f_{\Theta}(\theta)\, d\theta = \int_0^{2\pi} 5\cos(7t+\theta) \frac{1}{2\pi}\, d\theta = 0$



(b) Consider another random variable Y defined by

$$Y = 5\cos(7t_1 + \Theta) \times 5\cos(7t_2 + \Theta) = g(\Theta)$$

where t_1 and t_2 are some constants. Find $\mathbb{E}[Y]$.

Problem 5. A random variable X is a Gaussian random variable if its pdf is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2},$$

for some constant m and positive number σ. Furthermore, when a Gaussian random variable has $m = 0$ and $\sigma = 1$, we say that it is a standard Gaussian random variable. There is no closed-form expression for the cdf of the standard Gaussian random variable. The cdf itself is denoted by Φ and its values (or its complementary values $Q(\cdot) = 1 - \Phi(\cdot)$) are traditionally provided by a table.

Suppose Z is a standard Gaussian random variable. $f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$

$\Phi(z) = F_Z(z)$

(a) Use the Φ table to find the following probabilities:

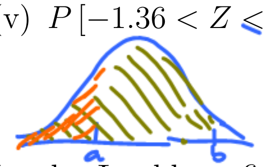
(i) $P[Z < 1.52]$ $= P[Z \le 1.52] = F_Z(1.52) = \Phi(1.52) \approx 0.9357$

(ii) $P[Z < -1.52]$ $= P[Z \le -1.52] = F_Z(-1.52) = \Phi(-1.52) = 1 - \Phi(1.52)$
$\approx 1 - 0.9357 = 0.0643$

(iii) $P[Z > 1.52]$

(iv) $P[Z > -1.52]$

(v) $P[\underset{a}{-1.36} < Z \le \underset{b}{1.52}]$ $= F_Z(b) - F_Z(a) = \Phi(b) - \Phi(a) = \Phi(b) - (1 - \Phi(a))$
$= \Phi(1.52) - (1 - \Phi(1.36)) \approx \ldots$



(b) Use the Φ table to find the value of c that satisfies each of the following relation.

(i) $P[Z > c] = 0.14$

(ii) $P[-c < Z < c] = 0.95$

Problem 6. The peak temperature T, as measured in degrees Fahrenheit, on a July day in New Jersey is a $\mathcal{N}(85, 100)$ random variable.

Remark: Do not forget that, for our class, the second parameter in $\mathcal{N}(\cdot, \cdot)$ is the variance (not the standard deviation).

(a) Express the cdf of T in terms of the Φ function.

(b) Express each of the following probabilities in terms of the Φ function(s). Make sure that the arguments of the Φ functions are positive. (Positivity is required so that we can directly use the Φ/Q tables to evaluate the probabilities.)

(i) $P[T > 100]$

(ii) $P[T < 60]$

(iii) $P[70 \le T \le 100]$

(c) Express each of the probabilities in part (b) in terms of the Q function(s). Again, make sure that the arguments of the Q functions are positive.

(i) $P[T > 100]$

(ii) $P[T < 60]$

(iii) $P[70 \le T \le 100]$

(d) Evaluate each of the probabilities in part (b) using the Φ/Q tables.

(i) $P[T > 100]$

(ii) $P[T < 60]$

(iii) $P[70 \leq T \leq 100]$

(e) Observe that the Φ table ("Table 4" from the lecture) stops at $z = 2.99$ and the Q table ("Table 5" from the lecture) starts at $z = 3.00$. Why is it better to give a table for $Q(z)$ instead of $\Phi(z)$ when z is large?

Problem 7. Suppose that the time to failure (in hours) of fans in a personal computer can be modeled by an exponential distribution with $\lambda = 0.0003$.

(a) What proportion of the fans will last at least 10,000 hours?

(b) What proportion of the fans will last at most 7000 hours?

[Montgomery and Runger, 2010, Q4-97]